Three Pillars of First Grade Mathematics

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For most students, first grade is the beginning of dedicated intensive instruction in mathematics. Since mathematics is a cumulative subject, getting started right is important. Since arithmetic is the main focus of mathematics education in elementary school, first grade should concentrate on giving students a good start in arithmetic. Some would argue that geometry or data or early algebra should also get attention, and there is probably room time to do something about some of these, but starting arithmetic off right is the essential task of first grade. Moreover, a good foundation in arithmetic can support learning both of data and algebra.

This is not as simple as it might sound. Getting going in arithmetic involves more than learning how to compute. It entails developing a sufficiently broad conception of the operations of addition and subtraction, which requires becoming acquainted with a wide range of situations where addition and subtraction can be used. It also entails going beyond situations that are described by counting, to see how arithmetic applies to the arena of measurement. The connection of arithmetic to geometry through measurement both enlarges the conception of arithmetic, and provides concrete and conceptual tools to help students think about arithmetic.

In the domain of computation, the overarching idea is that of place value. The standard conception of place value in the US is rather limited: we treat it as a vocabulary issue, that students should know the value of each place in a multi-digit number. However, the principle of place value controls essentially all aspects of arithmetic computation and estimation. Students should eventually come to realize and be able to exploit the ubiquitous influence of place value. A good start in first grade can make it easier for students to reach that goal.

These considerations lead to three main ingredients that are key to starting off right in arithmetic. They are:

I) A robust understanding of the operations of addition and subtraction.

II) An approach to arithmetic computation that intertwines place value with the addition/subtraction facts.

III) Making connections between counting number and measurement number.

Below we enlarge on each of these topics. A fourth section compares this plan with the Common Core Standards.

I. *A robust understanding of the operations of addition and subtraction.*

Addition is often described as combining, and subtraction as taking away, but the types of situations in which these operations are used are more varied than these brief descriptions would suggest. Mathematics educators have developed a taxonomy of one-step addition and subtraction word problems that recognizes 14 types of problem.

The types fall into three main categories: *change*, in which some number changes over time; *part-part whole*, in which some quantity or collection of objects is made up of two parts; and *comparison*, in which the difference between two quantities plays a role.

The first and third types can be subdivided in two. In problems involving change over time, the initial quantity can either increase or decrease. Similarly, in comparisons of quantities, one quantity can be described either as more or less than the other one. In part-part whole problems, the two parts play equivalent roles, so these form just one family.

Finally, for each of the four subtypes of change or comparison problems, one can pose three different questions, according as to what is unknown. Thus, for change-increase problems, one can ask to find the final total, the amount of change, or the initial amount. For comparison problems, one can ask to find the larger quantity, the smaller quantity, or the difference. For part-part-whole problems, since the two parts play equivalent roles, there are only two questions: what is the size of the whole, or what is the size of an unknown part. Thus there are four triples of problems (two triples involving change, two triples involving comparison), and one pair (of part-part-whole problems). In all, this gives $(2 \times 2 \times 3) + (1 \times 2) = 14$ types.

Here are examples of selected types:

Change-increase, total unknown: Shaquana had three toy trucks. For her birthday, she got four more toy trucks. How many toy trucks did she have then?

Comparison-more, smaller unknown: Shaquana has seven toy trucks. She has four more toy trucks than her friend Mollificent. How many toy trucks does Mollificent have?

Part-part whole, part unknown: Shaquana has a collection of seven toy trucks. She keeps them on two shelves in her bedroom. There are four trucks on the top shelf. How many trucks are on the lower shelf?

The full taxonomy, with all 14 subtypes (plus a fifteenth, of a different nature), is given as table I on page 88 of the Common Core State Standards (http://www.corestandards.org/the-standards/mathematics). Although an adult may think of these types of problem as quite similar, mathematics educators have shown that young children find them quite different [ ]. For example, consider the problem

Change-increase, original amount unknown: Shaquana had some toy trucks. For her birthday, she got four more toy trucks, and then she had seven. How many toy trucks did she have before her birthday?

This type of problem turns out to be difficult for young students to think about, because they are unsure how to model it. To solve the **Change-increase, total unknown** problem, they can count out three tokens, then four more tokens, then count all the tokens to find the answer. To deal with the **Change-increase, change unknown**, they can proceed similarly after some thought. They lay out seven counters to represent the total, and three next to them to represent the original amount. Then they count the unmatched counters in the total. (Effectively, they have converted the change problem to a comparison problem.) However, with the **Change-increase, original amount unknown**, they have trouble getting started. At this stage, the fact that a sum does not depend on the order in which the addends are combined (the *commutative* property of addition), is still to be learned.

The importance of presenting all types of addition and subtraction problems is clear if we take into account that a tremendous amount of learning takes place through examples. Children acquire vocabulary at the rate of about 8 words each day (see http://en.wikipedia.org/wiki/Vocabulary). Mostly, they do not look them up in the dictionary! Rather, they learn them by seeing them used in context, that is, through examples of how a word is used. Especially in teaching abstract concepts, which are the main content of mathematics, it is important to obey the maxim of *example sufficiency*, by which I mean, giving a broad enough array of examples to provide a well-rounded representation of the concept. A famous example of example insufficiency is the case of triangles. In brief presentations of the concept of triangle, frequently only one example, that of an equilateral triangle with a horizontal base, is given. Perhaps then it should not be surprising that studies have found that many second or third grade students will not identify non-equilateral triangles, or even equilateral triangles with non-horizontal bases, as being triangles. With foundational concepts, such as addition and subtraction, which will form the base on which many further ideas are built, it is especially important to present a well-rounded collection of situations where addition or subtraction can be used. Thus, care should be taken in first grade to introduce all types of one-step addition/subtraction word problems, and to use them all repeatedly throughout the year with larger numbers as student technique in symbolic calculation improves.

Sometimes, the use of only a limited number of the simplest problem types is justified on the basis that young students have limited reading skills, and that mathematics must be presented in ways that they can understand. This point of view might seem to have increased validity today, when so many students are classified as ELL (English-language learners). However, I would argue that mathematics word problems are as important for their potential to improve reading skills and thinking skills as they are for teaching mathematics. In fact, word problems are the glue that holds mathematics to the real world, and studying them from a language arts point of view, as passages that we want to understand, is as important as solving them. In doing this, comparative analysis may be an effective tool. Comparing and contrasting pairs of problems, then discussing all three of one of the triples of problems (e.g., the comparison-more triple), and ending with comparison of pairs of triples, may give students a sense for the territory of addition and subtraction in a way that just solving problems one at a time could not achieve. A somewhat subtle side benefit of this kind of activity may be, that some students come to think of addition and subtraction as having an existence independent of calculation, that is, they may realize that the expression $3 + 8$ is a valid name for a number whether or not we calculate to find that it is 11. This kind of understanding supports algebra.

An important part of discussing word problems, both individually and comparatively, is translating them into equations. Problems calling for subtraction can be represented as subtraction problems, and also as missing addend problems. Presenting both representations, and discussing their relationship can help establish the relationship between addition and subtraction. In writing equations, mathematical issues, such as the independence or order in addition, will probably arise spontaneously, and should be discussed as they do. Part-part-whole problems are the context where independence of order appears most naturally. Transferring this to the context of change problems can help children understand that the start unknown and change unknown problems are equivalent on a formal level.

Comparison problems require a special note of caution. In almost all uses of number that occur in everyday life, numbers function as adjectives: two hats, or two dollars, or two train rides all can be interpreted readily; however, "two" by itself does not have a clear meaning. Without a unit to refer to, the meaning of "two" is incomplete. Correspondingly, when we discuss addition, we understand (usually tacitly) that the numbers we are adding all refer to the same unit. The statement "3 dimes and 4 nickels equals 2 quarters" is perfectly intelligible. However, the equation $3 + 4 = 2$ violates our usual understandings of arithmetic. The source of the problem is that each number is referring to a different unit. To write an equation that expresses the desired relationship, we should make sure that all terms are denominated in the same unit. For example, if we express each coin, nickel, dime and quarter in terms of their value in pennies, we can write a correct equation: $3 \times 10 + 4 \times 5 = 2 \times 25$.

Since ignoring the unit usually does not cause trouble when dealing with whole numbers, units may often be suppressed in first grade and second grade texts. This can even serve a positive purpose, by emphasizing that arithmetic is independent of the unit: 4 apples and 3 apples make 7 apples, and likewise, 4 trucks and 3 trucks make 7 trucks. However, lack of unit awareness can wreak havoc in the study of fractions.

Comparison problems may seem to violate the same-unit principle. In such problems, one is often asked to compare the number of birds with the number of worms, or the number of children with the number of cookies. It may then seem that we are subtracting birds from worms, or the other way around, in contravention of the consistent unit principle. What is going on in these problems is more complicated. The problem scenario implicitly sets up a correspondence between the two sorts of things being compared, at some rate (often one-to–one). This implicit correspondence converts (implicitly, of course!) one of the quantities to the other, and subtraction takes place among the quantities of the type that is in abundance. However, this under-the-table correspondence may well be too subtle or confusing for young students to grasp. For this reason, it is advisable to formulate comparison problems so that they are about quantities of essentially the same type. For example, it is easier to assimilate "green apples" and "red apples" under the umbrella unit "apple" than it is to think of "tickets" and "people" as being essentially the same. Note that in the comparison example given above, all numbers referred to toy trucks.

II. *An approach to arithmetic computation that intertwines place value with the addition/subtraction facts.*

Place value is the central concept of arithmetic computation. It is not simply a vocabulary issue, of knowing the ones place, the tens place, and so on; it is the key organizing principle by which we deal with numbers. Place value together with the Rules of Arithmetic specifies the key aspects of how we perform addition/subtraction and multiplication/division (i.e., the algorithms of arithmetic). The vital role of place value is attested to by this quotation from Carl Friedrich Gauss (1777 - 1855), generally considered the greatest mathematician since Newton:

> The greatest calamity in the history of science
> was the failure of Archimedes to invent positional notation.

Two-digit numbers, and their addition and subtraction, is the topic where students first engage seriously with place value. The main ingredients in learning two-digit addition and subtraction are:

a) learning the addition/subtraction facts: knowing the sum of any two digits (meaning, the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), and, given the sum and one of the digits, knowing the other digit;

b) understanding that a two-digit number is made of some tens and some ones; and

c) in adding or subtracting, you work separately with the tens and the ones, <u>except</u> when regrouping is needed.

Specifically, item c) comprises two situations:

i) in adding, when you get more than 10 ones, you convert 10 of them into a ten,
and combine that with the other tens; or

ii) in subtracting, if the ones digit you want to subtract is larger than the ones digit
you want to subtract from, you must convert a ten into 10 ones,
and subtract from the resulting teen number.

The main US method for teaching this topic has been

a) learn the addition/subtraction facts by memorization; and

b) learn the column-wise algorithm for performing the operations.

These are typically treated separately, with little or no rationale given for either, and no connections between the two. In recent years, increased use of base ten blocks has probably increased some students' understanding of the regrouping process. However, the learning of the addition facts remains primarily a memorization process, unconnected with the other parts of the package, in particular, with regrouping. However, it is possible to combine the two key steps in such a way that they support each other, and are both connected to the fundamental principle of place value. This approach is widely used in East Asia. We will outline the main steps.

1) Learn the addition and subtraction facts to 10. This learning should be fluent, robust and flexible. This means understanding that 3 + 4 = 7, and 7 - 4 = 3, and also, that, if you have 4 and want 7, you need 3; and being able to produce any of these statements more or less automatically. Instruction should be accompanied by many concrete and pictorial illustrations of the relationships involved.

In learning the facts to 10, it is valuable to spend time thinking about all the possible ways to decompose a given number, for example to note that 5 = 4 + 1 = 3 + 2 = 2 + 3 = 1 + 4. Besides improving fluency, this work highlights structural facts, such as the commutative rule for addition, which reveals itself here in the symmetry of the possible expansions of 5: each decomposition is paired with a second one, in which the addends are in the opposite order. (Problems that call for all the possible ways to decompose a whole number into two smaller whole numbers are recognized as a fifteenth type of addition and subtraction problem in the Table I of the Common Core Standards, as cited above.)

2) Learn the teen numbers as a 10 and some ones. In Chinese, this is quite easy, because the number names express this directly: ten and one, ten and two, ten and three, and so on, up to two tens, and continue. It will involve more work in the US, since the number names are not as helpful. There will have to be class discussion about hearing the 10 in "teen", and hearing the 3 in "thir", so that students can think "ten and three" when they hear "thirteen". Similar work will have to be done with the other teen numbers. There will probably have to be some special talk about how "eleven" and "twelve" are pretty dumb names, but you just have to live with them, and think "ten and one" quietly to yourself when you hear "eleven".

Besides the names, the notation will need explicit attention. The fact that the 1 in 13 stands for ten, and the 3 stands for the three additional ones will probably have to be taken note of repeatedly. We agree not to write the 0 in the ten, to save time and space, but we put the 1 on the left of the three, and this is just a short way of writing 10 + 3. If our number names reinforced this, learning would probably be quicker and easier, but with sufficient reminders, we can hope that students will retain the idea.

Some amount of work with the teen numbers should be done to help students become comfortable with them. Asking which number comes just before or just after, asking which of two teen numbers is larger, and finally moving to addition or subtraction of single digit numbers to or from teen numbers (without regrouping), are examples of good exercises. Of course, some of these can be given as word problems. With regard to ordering, and adding that does not cross decades, students may observe spontaneously that only the ones digit is involved, and that, as far as this digit is concerned, everything is "just like" the parallel single digit behavior. If no student offers this, pointing it out may be helpful.

3) Learning the higher addition facts (addition and subtraction within 20): Now that the teen numbers are understood in terms of their base 10 structure, the focus returns to single-digit addition and subtraction, and learning the addition and subtraction facts when the total exceeds 10. Here the key point is not memorization of the higher addition facts, but understanding how to produce them, and their connection to place value notation. So, for example, to add 6 + 7, a student should think in terms of making a 10. From stage 1), it is known that starting from 6, one needs 4 more to make 10. One gets the 4 from the 7, and since

one also knows that 4 + 3 = 7, there are 3 left over from the 7, so one gets 10 and 3 more, or 13. In terms of symbolic manipulation, one has used the Associative Rule of addition to change the form of the sum:

$$6 + 7 = 6 + (4 + 3) = (6 + 4) + 3 = 10 + 3 = 13.$$

Similarly, in subtracting a one-digit number from a two-digit number, one may have to unmake or break apart the 10. There are (at least) two different ways that a student might think about this; either one is valid. These are illustrated in the following computations.

$$13 - 7 = 13 - (3 + 4) = (13 - 3) - 4 = 10 - 4 = 6,$$

or

$$13 - 7 = (10 + 3) - 7 = (10 - 7) + 3 = 3 + 3 = 6.$$

4) Learn that two-digit numbers are made of some tens and some ones: When students are fairly fluent in the addition/subtraction facts and making/unmaking 10, attention can move to larger numbers. The key understanding is that a two-digit number is made of some tens and some ones. Thus, 43 = 40 + 3 is 4 tens and 3 ones. The main work is probably in getting students to think of each -ty number as indicating a certain number of tens. Then the general two digit number is gotten by appending some ones, and this is fairly clearly indicated in the name. Again students need to learn to think beyond the names: "twenty" is 2 tens; "thirty" is 3 tens; "forty" is 4 tens; and so forth. The names and what they mean should again be connected with the notation: the 10s digit tells the number of tens, and the 1s digit tells the number of ones.

It is probably a good idea to do fair amount of counting of collections of objects to verify that two-digit number names mean what they do. As the counting is being done, the benefits of grouping by some manageable amount, which for us is 10, should be promoted. In fact, if counting gets interrupted, the advantage of having made groups of 10 should be evident, in greatly reducing the need for recounting. A hundreds chart can also be useful in this work. In working with a hundreds chart, it may be helpful to point out that a given number tells the number of spaces in the chart up to and including that number. This observation can also be helpful when studying computation (step 5 below), especially in interpreting the effect of adding 1 or adding 10 to a general two-digit number. Some educators advocate having a hundreds chart in which the numbers with a given tens digit run down a column (rather than across a row, which seems to be the more common form).

Manipulatives such as 10-rods and cubes may be helpful in making two-digit numbers tangible and accessible. Often such manipulatives are handled by arranging them in loose groupings, on a mat or other area designated for the work. However, it is probably a good idea to have students do some of this work in the context of linear measurement, with the 10-rods and cubes arranged into a linear train. Among other advantages, this will emphasize that the various rods and cubes are indeed united into a single quantity, with length corresponding to the size of the number. The measurement model for numbers is discussed further in section III below.

Attention should also be paid to ordering two-digit numbers – thinking about which of two numbers is larger. Here the simple principle is, that the 10s digit determines the relative size of two two-digit numbers, except when both numbers have the same 10s digit, in which case, you look at the ones digit. Since, the size difference between the 10-rods and the cubes is starkly apparent when all are assembled into a train, the measurement or length model of numbers, constructed by trains of 10-rods and cubes, can provide a physical and visual way of thinking about the relative sizes of numbers and the order relation.

5) Add/subtract two-digit numbers by combining tens with tens and ones with ones. This can be done in stages: add and subtract 1 or 10 from a two-digit number, add/subtract single digit numbers or multiples of ten from a two-digit number, add/subtract two-digit numbers without regrouping, add/subtract single digit numbers to or from two-digit numbers when regrouping is required, and finally, the general case of adding or subtracting two-digit numbers with regrouping. In adding (or subtracting) a single-digit number to (or from) a general two-digit number, when regrouping is required, the parallel with the corresponding addition fact should be emphasized. Both the reasoning and the mechanics of regrouping have already been learned while learning the addition facts beyond 10.

Manipulatives such as 10-rods and cubes can of course be used to model addition and subtraction. Again, arranging these rods and cubes into trains and working in terms of the length model for numbers can help students think about addition and subtraction. See section III for more details.

Ability to work independently with the tens and the ones should enable many students to do two-digit addition and subtraction mentally. To find 53 + 29, a student could say "50 + 20 is 70, and 3 + 9 is 12, and 70 + 12 is 82." To subtract 64 - 36, one could subtract 30 from 64 to get 34, which is 20 + 14, and then subtract 6 from 14 to get 28. (We note that this subtraction method, in which the largest place is subtracted first, will work in general. Of course it may involve more rewriting than the standard algorithm; but for two-digit numbers, it seems quite manageable.)

III. *Making connections between counting number and measurement number.*

One of the main arenas of application of mathematics is in measurement. Numbers used in measurement, in contrast to counting, may not be whole numbers. They can be fractions (technical term: *rational numbers*, meaning quotients of whole numbers (possibly also with a negative sign)), or even stranger numbers.* Geometrical measurement is so different from the context of counting, that the classical Greeks did not think of the numbers involved in measurement as numbers, and reserved the term *ratio* for numbers in the context of geometrical figures. It was only after the invention of symbolic algebra by Francois Viète around 1600, that the notion of number was expanded to include the numbers that arise in measurement. A few decades later, this development led to the invention of the coördinate plane by René Descartes, and to the strong linkage between number and geometry that we are used to today.

It is well to begin early helping students bridge the intuitive gap between number and geometry, and there are several benefits to starting in first grade. In particular, this can already help students think geometrically about two-digit numbers.

The most basic and probably simplest type of measurement is *linear measurement*: measurement of length or distance. Probably most adults think of linear measurement in terms of using a ruler. However, one should first lay a foundation for this by getting students to think of length or distance in terms of the familiar counting numbers, and to model addition by *concatenation of length* – laying rods end to end.

This process lends itself well to work with manipulatives in first grade. The basic materials needed are a collection of unit cubes, and rods with the same cross section as the cubes, but of various lengths. All whole number lengths from 1 to 20 would afford exploration of addition and subtraction within 20, in other words, a measurement analog of the addition and subtraction facts. Cuisenaire rods can probably be useful, but they don't have the full range of lengths, and their colors may be a distraction. Besides cubes, a generous supply of rods of length 10 is desirable.

A first activity would be just measuring the length of various rods in terms of the cubes. It might be a good exercise to see if students could learn to recognize various lengths without having to measure. The rods might be marked with their lengths to facilitate later work (or if Cuisenaire rods are being used, many students will probably learn to associate lengths with the colors). In learning to measure, students should come to appreciate the importance of lining up the cubes carefully, face to face, with no gaps.

After students have gained familiarity with measuring the rods, and have come to associate a definite length with a given rod, along with associated ideas of order – that longer rods have greater measured lengths – addition and subtraction can be studied. Students should get used to the idea that addition corresponds to putting bars together end-to-end, aka combination of lengths. Subtraction corresponds to comparison of lengths: placing two rods side-by-side, and measuring the unmatched part of the longer rod. After a reasonable amount of work like this, the reasons for these correspondences between length measurement and arithmetic should be discussed, Ideally, a student will volunteer the basic reason: we have defined length in terms of measurement by unit lengths, and the collection of units needed to measure a combination of lengths is just the union of the collections that measure each of the individual lengths. Similar reasoning applies to subtraction.

Once addition and subtraction are interpreted in terms of lengths, one can begin to use the length model to bolster understanding of place value. One can introduce 10-rods as a convenient way to simplify

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* For example, the number π, which is involved in the formulas for the circumference and the area of a circle of given radius, is an example of what is usually called an *irrational* number, meaning a number that is not a quotient of whole numbers.

the measuring process. The ease of laying down one 10-rod instead of carefully lining up ten units cubes should be apparent to students. The expression of the teen numbers as a 10 and some 1s is readily modeled with a 10-rod and cubes, and the modeling of the addition and subtraction facts beyond 10, and the making (in addition) and unmaking (in subtraction) of a 10 can be illustrated concretely in terms of length.

At some point, the possibility of measuring other lengths – lengths of pencils, length and widths of book covers, various body parts, and anything else that attracts class attention – should be explored. Longer things can be measured as students become accustomed to dealing with larger numbers. At least some measurement should be done using unit cubes only, so that the huge savings in effort afforded by use of 10-rods instead of only using unit cubes is made evident. Reporting of results of measurement should include units – so many cubes long. If the cube sides are of a standard length, such as a centimeter, this term could be used. Whether it is necessary or advisable at this point to consider different units of length needs study.

Objects in the class environment will typically not be exactly whole numbers of units in length. Often it is advocated to have students say that a given object is "about 14" units long. However, I would favor reporting the length as "between 14 and 15" (if it is more than 14; or "between 13 and 14" if less). This kind of language serves to highlight the need for more numbers than whole numbers in the realm of measurement. Indeed, a teacher could tell students that later they will learn about other numbers (fractions, mixed numbers, rational numbers) that can be used to measure more accurately. The value of the length model for addition and subtraction will be fully realized when these numbers are the focus of study, because although the symbolic representation of addition is substantially more complicated for fractions than for whole numbers, the geometric representation in terms of combination of lengths is uniform throughout.

When students are used to thinking of addition in terms of combining lengths, and familiar with 10-rods, the length model for addition can be coordinated with base 10 notation. Students can make trains consisting of 10-rods and cubes, to represent two-digit numbers. The convention should be established that the standard way to do this is always to have the 10-rods together one one side of the train (say the left), and the cubes together on the other (the right). This arrangement best displays the base ten structure of the number.

When numbers so represented are added by combining the trains end-to-end, students will probably observe that the resulting train is not in standard form: the 10-rods of the train on the right are to the right of the cubes of the train on the left. To put the train in standard order, these rods and cubes must be rearranged. The resulting train will be seen, perhaps after sufficient teacher direction, to be the result of "combining the tens and combining the ones", just as in the other contexts where two-digit addition is studied. Also, if the sum has more than ten cubes, the regrouping process can be modeled physically by replacing ten of the cubes by one 10-rod. If students fail to do so, it probably should be explicitly noted by the teacher that this process preserves the total length.

The analog of this process for subtraction should also be done carefully. When no regrouping is required, the trains of 10-rods can be compared to each other, and the trains of cubes likewise, and it should be checked that this yields the same answer as full train comparison. When regrouping is required, one can literally borrow one 10-rod and convert it to cubes, to supplement the cubes in the minuend, before comparing with the cubes in the subtrahend. As with addition, the results of the separate comparison processes for the 10-rods and the cubes should be verified to give the same result as the whole train comparison.

This kind of work with lengths can strengthen the learning of arithmetic by reinforcing symbolic work and work with unstructured collections of objects. Equally important, it should get children used to the idea that measurement is a natural domain for application of number ideas. It should prepare them well for introduction of the number line (more correctly speaking, number *ray*), whose concrete realization is the ruler, as a tool that can be used measure anything without the need to form trains at all, in second grade.

IV. *Relation to the Common Core Standards.*

At this moment in the US, any set of recommendations about mathematics instruction can not hope to garner serious attention without taking into account the Common Core State Standards, developed in 2009-10 by the Council of Chief State School Officers and the National Governor's Association, and since adopted by over 40 states. So we briefly compare the program advocated here with the First Grade Common Core State Standards.

Overall, the recommendations above are highly compatible with CCSS. It seems fair to describe the ideas in this note as a more focused, and more connected, and thereby, we hope, a more coherent, version of the CCSS. The first CCSS standard on addition and subtraction mentions all the problem types discussed in section I, and Table I on page 88 gives the full taxonomy. There is less emphasis on understanding all the problems and on comparative discussions, in CCSS, but such discussions can be regarded as the concrete realization in the first grade context of some of the standards for Mathematical Practice, including Practice Standard 1:"Make sense of problems and persevere in solving them."; Practice Standard 4: "Model with mathematics."; and Practice Standard 6:"Attend to precision." The standards 3 and 4, and 7 and 8 under "Operations and Algebraic Thinking" should all be part of discussing word problems and representing them symbolically.

Part II encompasses the standards 5 and 6 about "Addition and Subtraction within 20" under "Operations and Algebraic Thinking", the standards 2 and 3 about "Understand lace value" under "Number and Operations in Base Ten", and also the standards 4, 5 and 6 about "Use place value understanding and properties of operations to add and subtract". This section describes how to combine these standards into the connected and coherent teaching sequence that is widely used in East Asia to teach two-digit addition and subtraction. Standard 4 states the principle of "add the tens, and (separately) add the ones' (and sometimes compose a 10 from ones). Standards 5 and 6 can be thought of as pointing out special test cases that can be used as stepping stones to the full principle.

The standards on measurement of lengths (standards 1 and 2 under "Measurement and Data") can be thought of as part of what is advocated in section III. Section III puts more emphasis on connecting this work with the numerical work. In doing this, it helps prepare for introduction of the number line in second grade, and creates a more cohesive overall package.

The CCSS has some standards which are not related to the program discussed here. There are standards on time, and on representing data, and on studying shapes. As noted in the introduction, there is probably time to do some of these things during first grade. The import of this note, however, is that the three topics highlighted here form the core of instruction, the part that is essential for future work, and it advocates for weaving them together into a coherent whole.