Progressions for the Common Core State Standards in Mathematics (draft)

21 April 2012

K–5, Number and Operations in Base Ten

Overview

Students' work in the base-ten system is intertwined with their work on counting and cardinality, and with the meanings and properties of addition, subtraction, multiplication, and division. Work in the base-ten system relies on these meanings and properties, but also contributes to deepening students' understanding of them.

Position The base-ten system is a remarkably efficient and uniform system for systematically representing all numbers. Using only the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, every number can be represented as a string of digits, where each digit represents a value that depends on its place in the string. The relationship between values represented by the places in the base-ten system is the same for whole numbers and decimals: the value represented by each place is always 10 times the value represented by the place to its immediate right. In other words, moving one place to the left, the value of the place is multiplied by 10. In moving one place to the right, the value of the place is divided by 10. Because of this uniformity, standard algorithms for computations within the base-ten system for whole numbers extend to decimals.

Base-ten units Each place of a base-ten numeral represents a base-ten unit: ones, tens, tenths, hundreds, hundredths, etc. The digit in the place represents 0 to 9 of those units. Because ten like units make a unit of the next highest value, only ten digits are needed to represent any quantity in base ten. The basic unit is a *one* (represented by the rightmost place for whole numbers). In learning about whole numbers, children learn that ten ones compose a new kind of unit called a *ten*. They understand two-digit numbers as composed of tens and ones, and use this understanding in computations, decomposing 1 ten into 10 ones and composing a ten from 10 ones.

The power of the base-ten system is in repeated bundling by ten: 10 tens make a unit called a hundred. Repeating this process of creating new units by bundling in groups of ten creates units called *thousand*, *ten thousand*, *hundred thousand* .... In learning about decimals, children partition a one into 10 equal-sized smaller units, each of which is a tenth. Each base-ten unit can be understood in terms of any other base-ten unit. For example, one hundred can be viewed as a tenth of a thousand, 10 tens, 100 ones, or 1,000 tenths. Algorithms for operations in base ten draw on such relationships among the base-ten units.

Computations Standard algorithms for base-ten computations with the four operations rely on decomposing numbers written in base-ten notation into base-ten units. The properties of operations then allow any multi-digit computation to be reduced to a collection of single-digit computations. These single-digit computations sometimes require the composition or decomposition of a base-ten unit.

Beginning in Kindergarten, the requisite abilities develop gradually over the grades. Experience with addition and subtraction within 20 is a Grade 1 standard[1.OA.6] and fluency is a Grade 2 standard.[2.OA.2] Computations within 20 that "cross 10," such as $9+8$ or $13-6$, are especially relevant to NBT because they afford the development of the Level 3 make-a-ten strategies for addition and subtraction described in the OA Progression. From the NBT perspective, make-a-ten strategies are (implicitly) the first instances of composing or decomposing a base-ten unit. Such strategies are a foundation for understanding in Grade 1 that addition may require composing a ten[1.NBT.4] and in Grade 2 that subtraction may involve decomposing a ten.[2.NBT.7]

Strategies and algorithms The Standards distinguish strategies from algorithms.• For example, students use strategies for addition and subtraction in Grades K-3, but are expected to fluently add and subtract whole numbers using standard algorithms by the end of Grade 4. Use of the standard algorithms can be viewed as the culmination of a long progression of reasoning about quantities, the base-ten system, and the properties of operations.

This progression distinguishes between two types of computational strategies: special strategies and general methods. For example, a special strategy for computing $398+17$ is to decompose 17 as $2+15$, and evaluate $(398+2)+15$. Special strategies either cannot be extended to all numbers represented in the base-ten system or require considerable modification in order to do so. A more readily generalizable method of computing $398+17$ is to combine like base-ten units. General methods extend to all numbers represented in the base-ten system. A general method is not necessarily efficient. For example, counting on by ones is a general method that can be easily modified for use with finite decimals. General methods based on place value, however, are more efficient and can be viewed as closely connected with standard algorithms.

1.OA.6 Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., $8+6=8+2+4=10+4=14$); decomposing a number leading to a ten (e.g., $13-4=13-3-1=10-1=9$); using the relationship between addition and subtraction (e.g., knowing that $8+4=12$, one knows $12-8=4$); and creating equivalent but easier or known sums (e.g., adding $6+7$ by creating the known equivalent $6+6+1=12+1=13$).

2.OA.2 Fluently add and subtract within 20 using mental strategies.[1] By end of Grade 2, know from memory all sums of two one-digit numbers.

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

• **Computation algorithm.** A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. See also: computation strategy.
Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. See also: computation algorithm.

Mathematical practices Both general methods and special strategies are opportunities to develop competencies relevant to the NBT standards. Use and discussion of both types of strategies offer opportunities for developing fluency with place value and properties of operations, and to use these in justifying the correctness of computations (MP.3). Special strategies may be advantageous in situations that require quick computation, but less so when uniformity is useful. Thus, they offer opportunities to raise the topic of using appropriate tools strategically (MP.5). Standard algorithms can be viewed as expressions of regularity in repeated reasoning (MP.8) used in general methods based on place value.

Numerical expressions and recordings of computations, whether with strategies or standard algorithms, afford opportunities for students to contextualize, probing into the referents for the symbols involved (MP.2). Representations such as bundled objects or math drawings (e.g., drawings of hundreds, tens, and ones) and diagrams (e.g., simplified renderings of arrays or area models) afford the mathematical practice of explaining correspondences among different representations (MP.1). Drawings, diagrams, and numerical recordings may raise questions related to precision (MP.6), e.g., does that 1 represent 1 one or 1 ten? This progression gives examples of representations that can be used to connect numerals with quantities and to connect numerical representations with combination, composition, and decomposition of base-ten units as students work towards computational fluency.

Kindergarten

In Kindergarten, teachers help children lay the foundation for understanding the base-ten system by drawing special attention to 10. Children learn to view the whole numbers 11 through 19 as ten ones and some more ones. They decompose 10 into pairs such as $1 + 9$, $2+8$, $3+7$ and find the number that makes 10 when added to a given number such as 3 (see the OA Progression for further discussion).

Work with numbers from 11 to 19 to gain foundations for place value[K.NBT.1] Children use objects, math drawings,• and equations to describe, explore, and explain how the "teen numbers," the counting numbers from 11 through 19, are ten ones and some more ones. Children can count out a given teen number of objects, e.g., 12, and group the objects to see the ten ones and the two ones. It is also helpful to structure the ten ones into patterns that can be seen as ten objects, such as two fives (see the OA Progression).

A difficulty in the English-speaking world is that the words for teen numbers do not make their base-ten meanings evident. For example, "eleven" and "twelve" do not sound like "ten and one" and "ten and two." The numbers "thirteen, fourteen, fifteen, . . . , nineteen" reverse the order of the ones and tens digits by saying the ones digit first. Also, "teen" must be interpreted as meaning "ten" and the prefixes "thir" and "fif" do not clearly say "three" and "five." In contrast, the corresponding East Asian number words are "ten one, ten two, ten three," and so on, fitting directly with the base-ten structure and drawing attention to the role of ten. Children could learn to say numbers in this East Asian way in addition to learning the standard English number names. Difficulties with number words beyond nineteen are discussed in the Grade 1 section.

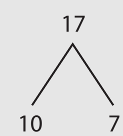
The numerals $11, 12, 13, \ldots, 19$ need special attention for children to understand them. The first nine numerals $1, 2, 3, \ldots, 9$, and 0 are essentially arbitrary marks. These same marks are used again to represent larger numbers. Children need to learn the differences in the ways these marks are used. For example, initially, a numeral such as 16 looks like "one, six," not "1 ten and 6 ones ." Layered place value cards can help children see the 0 "hiding" under the ones place and that the 1 in the tens place really is 10 (ten ones).

By working with teen numbers in this way in Kindergarten, students gain a foundation for viewing 10 ones as a new unit called a ten in Grade 1.

K.NBT.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 + 8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

• Math drawings are simple drawings that make essential mathematical features and relationships salient while suppressing details that are not relevant to the mathematical ideas.

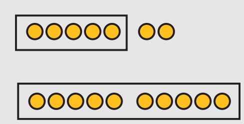
Number-bond diagram and equation



$17 = 10 + 7$

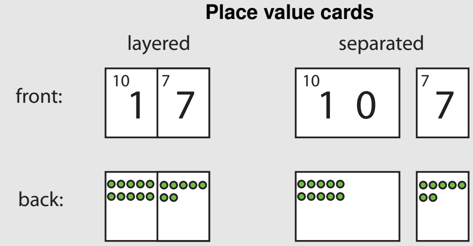
Decompositions of teen numbers can be recorded with diagrams or equations.

5- and 10-frames



Children can place small objects into 10-frames to show the ten as two rows of five and the extra ones within the next 10-frame, or work with strips that show ten ones in a column.

Place value cards



Children can use layered place value cards to see the 10 "hiding" inside any teen number. Such decompositions can be connected to numbers represented with objects and math drawings.

Grade 1

In first grade, students learn to view ten ones as a unit called a ten. The ability to compose and decompose this unit flexibly and to view the numbers 11 to 19 as composed of one ten and some ones allows development of efficient, general base-ten methods for addition and subtraction. Students see a two-digit numeral as representing some tens and they add and subtract using this understanding.

Extend the counting sequence and understand place value Through practice and structured learning time, students learn patterns in spoken number words and in written numerals, and how the two are related.

Grade 1 students take the important step of viewing ten ones as a unit called a "ten."[1.NBT.2a] They learn to view the numbers 11 through 19 as composed of 1 ten and some ones.[1.NBT.2b] They learn to view the decade numbers $10, \ldots, 90$, in written and in spoken form, as 1 ten, ..., 9 tens.[1.NBT.2c] More generally, first graders learn that the two digits of a two-digit number represent amounts of tens and ones, e.g., 67 represents 6 tens and 7 ones.

1.NBT.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

a 10 can be thought of as a bundle of ten ones—called a "ten."

b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).

The number words continue to require attention at first grade because of their irregularities. The decade words, "twenty," "thirty," "forty," etc., must be understood as indicating 2 tens, 3 tens, 4 tens, etc. Many decade number words sound much like teen number words. For example, "fourteen" and "forty" sound very similar, as do "fifteen" and "fifty," and so on to "nineteen" and "ninety." As discussed in the Kindergarten section, the number words from 13 to 19 give the number of ones before the number of tens. From 20 to 100, the number words switch to agreement with written numerals by giving the number of tens first. Because the decade words do not clearly indicate they mean a number of tens ("-ty" does mean tens but not clearly so) and because the number words "eleven" and "twelve" do not cue students that they mean "1 ten and 1" and "1 ten and 2," children frequently make count errors such as "twenty-nine, twenty-ten, twenty-eleven, twenty-twelve."

Grade 1 students use their base-ten work to help them recognize that the digit in the tens place is more important for determining the size of a two-digit number.[1.NBT.3] They use this understanding to compare two two-digit numbers, indicating the result with the symbols $>$, $=$, and $<$. Correctly placing the $<$ and $>$ symbols is a challenge for early learners. Accuracy can improve if students think of putting the wide part of the symbol next to the larger number.

1.NBT.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols $>$, $=$, and $<$.

Use place value understanding and properties of operations to add and subtract First graders use their base-ten work to compute sums within 100 with understanding.[1.NBT.4] Concrete objects, cards, or drawings afford connections with written numerical work and discussions and explanations in terms of tens and ones. In particular, showing composition of a ten with objects or drawings

1.NBT.4 Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.

affords connection of the visual ten with the written numeral 1 that indicates 1 ten.

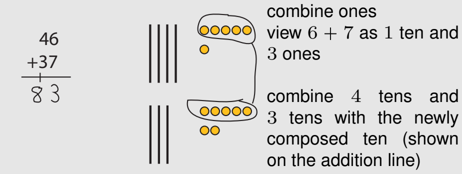
Adding tens and ones separately as illustrated in the margin is a general method that can extend to any sum of multi-digit numbers. Students may also develop sequence methods that extend their Level 2 single-digit counting on strategies (see the OA Progression) to counting on by tens and ones, or mixtures of such strategies in which they add instead of count the tens or ones. Using objects or drawings of 5-groups can support students' extension of the Level 3 make-a-ten methods discussed in the OA Progression for single-digit numbers.

First graders also engage in mental calculation, such as mentally finding 10 more or 10 less than a given two-digit number without having to count by ones.[1.NBT.5] They may explain their reasoning by saying that they have one more or one less ten than before. Drawings and layered cards can afford connections with place value and be used in explanations.

In Grade 1, children learn to compute differences of two-digit numbers for limited cases.[1.NBT.6] Differences of multiples of 10, such as $70 - 40$ can be viewed as 7 tens minus 4 tens and represented with concrete models such as objects bundled in tens or drawings. Children use the relationship between subtraction and addition when they view $80 - 70$ as an unknown addend addition problem, $70 + \square = 80$, and reason that 1 ten must be added to 70 to make 80, so $80 - 70 = 10$.

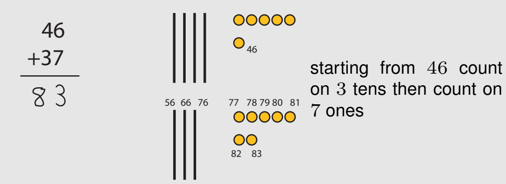
First graders are not expected to compute differences of two-digit numbers other than multiples of ten. Deferring such work until Grade 2 allows two-digit subtraction with and without decomposing to occur in close succession, highlighting the similarity between these two cases.

General method: Adding tens and ones separately



This method is an application of the associative property.

Special strategy: Counting on by tens



This strategy requires counting on by tens from 46, beginning 56, 66, 76, then counting on by ones.

1.NBT.5 Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

1.NBT.6 Subtract multiples of 10 in the range 10–90 from multiples of 10 in the range 10–90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Grade 2

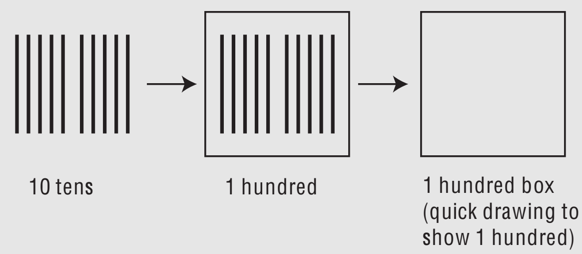
At Grade 2, students extend their base-ten understanding to hundreds. They now add and subtract within 1000, with composing and decomposing, and they understand and explain the reasoning of the processes they use. They become fluent with addition and subtraction within 100.

Understand place value In Grade 2, students extend their understanding of the base-ten system by viewing 10 tens as forming a new unit called a "hundred."[2.NBT.1a] This lays the groundwork for understanding the structure of the base-ten system as based in repeated bundling in groups of 10 and understanding that the unit associated with each place is 10 of the unit associated with the place to its right.

2.NBT.1a Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:

a 100 can be thought of as a bundle of ten tens—called a "hundred."

Drawings to support seeing 10 tens as 1 hundred



Representations such as manipulative materials, math drawings and layered three-digit place value cards afford connections between written three-digit numbers and hundreds, tens, and ones. Number words and numbers written in base-ten numerals and as sums of their base-ten units can be connected with representations in drawings and place value cards, and by saying numbers aloud and in terms of their base-ten units, e.g., 456 is "Four hundred fifty six" and "four hundreds five tens six ones."[2.NBT.3]

2.NBT.3 Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.

Unlike the decade words, the hundred words indicate base-ten units. For example, it takes interpretation to understand that "fifty" means five tens, but "five hundred" means almost what it says ("five hundred" rather than "five hundreds"). Even so, this doesn't mean that students automatically understand 500 as 5 hundreds; they may still only think of it as the number reached after 500 counts of 1.

Students begin to work towards multiplication when they skip count by 5s, by 10s, and by 100s. This skip counting is not yet true multiplication because students don't keep track of the number of groups they have counted.[2.NBT.2]

2.NBT.2 Count within 1000; skip-count by 5s, 10s, and 100s.

Comparing magnitudes of two-digit numbers draws on the understanding that 1 ten is greater than any amount of ones represented by a one-digit number. Comparing magnitudes of three-digit numbers draws on the understanding that 1 hundred (the smallest three-digit number) is greater than any amount of tens and ones represented by a two-digit number. For this reason, three-digit numbers are compared by first inspecting the hundreds place (e.g. $845 > 799$; $849 < 855$).[2.NBT.4]

2.NBT.4 Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, using $>$, =, and $<$ symbols to record the results of comparisons.

Use place value understanding and properties of operations to add and subtract Students become fluent in two-digit addition and subtraction.[2.NBT.5, 2.NBT.6] Representations such as manipulative materials and drawings may be used to support reasoning and explanations about addition and subtraction with three-digit numbers.[2.NBT.7] When students add ones to ones, tens to tens, and hundreds to hundreds they are implicitly using a general method based on place

2.NBT.5 Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.

2.NBT.6 Add up to four two-digit numbers using strategies based on place value and properties of operations.

2.NBT.7 Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.

value and the associative and commutative properties of addition. Examples of how general methods can be represented in numerical work and composition and decomposition can be represented in math drawings are shown in the margin.

Drawings and diagrams can illustrate the reasoning repeated in general methods for computation that are based on place value. These provide an opportunity for students to observe this regularity and build toward understanding the standard addition and subtraction algorithms required in Grade 4 as expressions of repeated reasoning (MP.8).

At Grade 2, composing and decomposing involves an extra layer of complexity beyond that of Grade 1. This complexity manifests itself in two ways. First, students must understand that a hundred is a unit composed of 100 ones, but also that it is composed of 10 tens. Second, there is the possibility that both a ten and a hundred are composed or decomposed. For example, in computing $398 + 7$ a new ten and a new hundred are composed. In computing $302 - 184$, a ten and a hundred are decomposed.

Students may continue to develop and use special strategies for particular numerical cases or particular problem situations such as Unknown Addend. For example, instead of using a general method to add $398 + 7$, students could reason mentally by decomposing the 7 ones as $2 + 5$, adding 2 ones to 398 to make 400, then adding the remaining 5 ones to make 405. This method uses the associative property of addition and extends the make-a-ten strategy described in the OA Progression. Or students could reason that 398 is close to 400, so the sum is close to $400 + 7$, which is 407, but this must be 2 too much because 400 is 2 more than 398, so the actual sum is 2 less than 407, which is 405. Both of these strategies make use of place value understanding and are practical in limited cases.

Subtractions such as $302 - 184$ can be computed using a general method by decomposing a hundred into 10 tens, then decomposing one of those tens into 10 ones. Students could also view it as an unknown addend problem $184 + \square = 302$, thus drawing on the relationship between subtraction and addition. With this view, students can solve the problem by adding on to 184: first add 6 to make 190, then add 10 to make 200, next add 100 to make 300, and finally add 2 to make 302. They can then combine what they added on to find the answer to the subtraction problem: $6 + 10 + 100 + 2 = 118$. This strategy is especially useful in unknown addend situations. It can be carried out more easily in writing because one does not have to keep track of everything mentally. This is a Level 3 strategy, and is easier than the Level 3 strategy illustrated below that requires keeping track of how much of the second addend has been added on. (See the OA Progression for further discussion of levels.)

When computing sums of three-digit numbers, students might use strategies based on a flexible combination of Level 3 composition and decomposition and Level 2 counting-on strategies when finding the value of an expression such as $148 + 473$. For exam-

Addition: Recording combined hundreds, tens, and ones on separate lines

```
  456        456        456        456
+ 167      + 167      + 167      + 167
-----      -----      -----      -----
             500        500        500
                        110        110
                                    13
                                   ---
                                   623
```

Addition proceeds from left to right, but could also have gone from right to left. There are two advantages of working left to right: Many students prefer it because they read from left to right, and working first with the largest units yields a closer approximation earlier.

Addition: Recording newly composed units on the same line

```
  456        4 5 6      456        456
+ 167      + 1 6 7    + 167      + 167
-----          1       1 1        1 1
               3      -----      -----
                        23        623
```

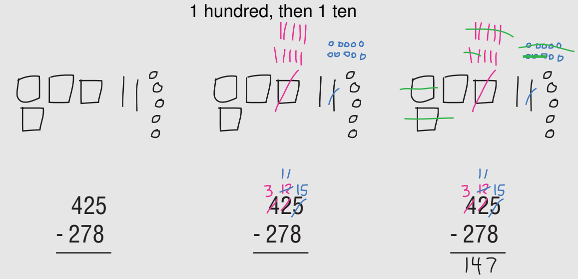
	Add the ones, $6 + 7$, and record these 13 ones with 3 in the ones place and 1 on the line under the tens column.	Add the tens, $5 + 6 + 1$, and record these 12 tens with 2 in the tens place and 1 on the line under the hundreds column.	Add the hundreds, $4 + 1 + 1$ and record these 6 hundreds in the hundreds column.

Digits representing newly composed units are placed below the addends. This placement has several advantages. Each two-digit partial sum (e.g., "13") is written with the digits close to each other, suggesting their origin. In "adding from the top down," usually sums of larger digits are computed first, and the easy-to-add "1" is added to that sum, freeing students from holding an altered digit in memory. The original numbers are not changed by adding numbers to the first addend; three multi-digit numbers (the addends and the total) can be seen clearly. It is easier to write teen numbers in their usual order (e.g., as 1 then 3) rather than "write the 3 and carry the 1" (write 3, then 1).

Subtraction: Decomposing where needed first

decomposing left to right, 1 hundred, then 1 ten

now subtract



All necessary decomposing is done first, then the subtractions are carried out. This highlights the two major steps involved and can help to inhibit the common error of subtracting a smaller digit on the top from a larger digit. Decomposing and subtracting can start from the left (as shown) or the right.

ple, they might say, "100 and 400 is 500. And 70 and 30 is another hundred, so 600. Then 8, 9, 10, 11 ... and the other 10 is 21. So, 621." Keeping track of what is being added is easier using a written form of such reasoning and makes it easier to discuss. There are two kinds of decompositions in this strategy. Both addends are decomposed into hundreds, tens, and ones, and the first addend is decomposed successively into the part already added and the part still to add.

Students should continue to develop proficiency with mental computation. They mentally add 10 or 100 to a given number between 100 and 900, and mentally subtract 10 or 100 from a given number between 100 and 900.[2.NBT.8]

2.NBT.8 Mentally add 10 or 100 to a given number 100–900, and mentally subtract 10 or 100 from a given number 100–900.

Grade 3

At Grade 3, the major focus is multiplication,• so students' work with addition and subtraction is limited to maintenance of fluency within 1000 for some students and building fluency to within 1000 for others.

• See the progression on Operations and Algebraic Thinking.

Use place value understanding and properties of operations to perform multi-digit arithmetic Students continue adding and subtracting within 1000.[3.NBT.2] They achieve fluency with strategies and algorithms that are based on place value, properties of operations, and/or the relationship between addition and subtraction. Such fluency can serve as preparation for learning standard algorithms in Grade 4, if the computational methods used can be connected with those algorithms.

3.NBT.2 Fluently add and subtract within 1000 using strategies and algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.

Students use their place value understanding to round numbers to the nearest 10 or 100.[3.NBT.1] They need to understand that when moving to the right across the places in a number (e.g., 456), the digits represent smaller units. When rounding to the nearest 10 or 100, the goal is to approximate the number by the closest number with no ones or no tens and ones (e.g., so 456 to the nearest ten is 460; and to the nearest hundred is 500). Rounding to the unit represented by the leftmost place is typically the sort of estimate that is easiest for students. Rounding to the unit represented by a place in the middle of a number may be more difficult for students (the surrounding digits are sometimes distracting). Rounding two numbers before computing can take as long as just computing their sum or difference.

3.NBT.1 Use place value understanding to round whole numbers to the nearest 10 or 100.

The special role of 10 in the base-ten system is important in understanding multiplication of one-digit numbers with multiples of 10.[3.NBT.3] For example, the product 3×50 can be represented as 3 groups of 5 tens, which is 15 tens, which is 150. This reasoning relies on the associative property of multiplication: $3 \times 50 = 3 \times (5 \times 10) = (3 \times 5) \times 10 = 15 \times 10 = 150$. It is an example of how to explain an instance of a calculation pattern for these products: calculate the product of the non-zero digits, then shift the product one place to the left to make the result ten times as large.•

3.NBT.3 Multiply one-digit whole numbers by multiples of 10 in the range 10–90 (e.g., 9×80, 5×60) using strategies based on place value and properties of operations.

• **Grade 3 explanations for "15 tens is 150"**

- Skip-counting by 50. 5 tens is 50, 100, 150.
- Counting on by 5 tens. 5 tens is 50, 5 more tens is 100, 5 more tens is 150.
- Decomposing 15 tens. 15 tens is 10 tens and 5 tens. 10 tens is 100. 5 tens is 50. So 15 tens is 100 and 50, or 150.
- Decomposing 15.

$$\begin{aligned} 15 \times 10 &= (10 + 5) \times 10 \\ &= (10 \times 10) + (5 \times 10) \\ &= 100 + 50 \\ &= 150 \end{aligned}$$

All of these explanations are correct. However, skip-counting and counting on become more difficult to use accurately as numbers become larger, e.g., in computing 5×90 or explaining why 45 tens is 450, and needs modification for products such as 4×90. The first does not indicate any place value understanding.

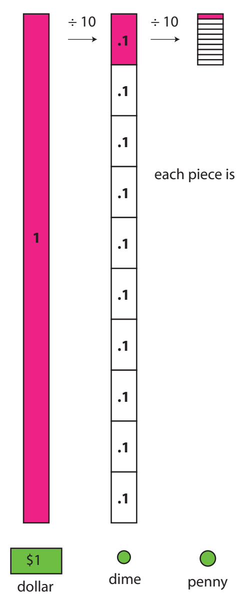
Grade 4

At Grade 4, students extend their work in the base-ten system. They use standard algorithms to fluently add and subtract. They use methods based on place value and properties of operations supported by suitable representations to multiply and divide with multi-digit numbers.

Generalize place value understanding for multi-digit whole numbers In the base-ten system, the value of each place is 10 times the value of the place to the immediate right.[4.NBT.1] Because of this, multiplying by 10 yields a product in which each digit of the multiplicand is shifted one place to the left.

To read numerals between 1,000 and 1,000,000, students need to understand the role of commas. Each sequence of three digits made by commas is read as hundreds, tens, and ones, followed by the name of the appropriate base-thousand unit (thousand, million, billion, trillion, etc.). Thus, 457,000 is read "four hundred fifty seven thousand."[4.NBT.2] The same methods students used for comparing and rounding numbers in previous grades apply to these numbers, because of the uniformity of the base-ten system.

Decimal notation and fractions Students in Grade 4 work with fractions having denominators 10 and 100.[4.NF.5] Because it involves partitioning into 10 equal parts and treating the parts as numbers called one tenth and one hundredth, work with these fractions can be used as preparation to extend the base-ten system to non-whole numbers.

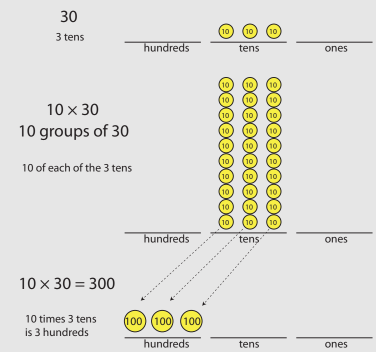


Using the unit fractions $\frac{1}{10}$ and $\frac{1}{100}$, non-whole numbers like $23\frac{7}{10}$ can be written in an expanded form that extends the form used with whole numbers: $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$.[4.NF.4b] As with whole-number expansions in the base-ten system, each unit in this decomposition is ten times the unit to its right. This can be connected with the use of base-ten notation to represent $2 \times 10 + 3 \times 1 + 7 \times \frac{1}{10}$ as 23.7. Using decimals allows students to apply familiar place value reasoning to fractional quantities.[4.NF.6] The Number and Operations—Fractions Progression discusses decimals to hundredths and comparison of decimals[4.NF.7] in more detail.

The decimal point is used to signify the location of the ones place, but its location may suggest there should be a "oneths" place to its right in order to create symmetry with respect to the decimal point.

4.NBT.1 Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.

10×30 represented as 3 tens each taken 10 times

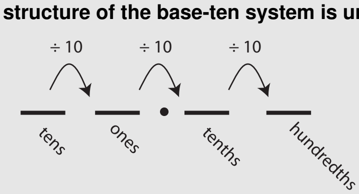


Each of the 3 tens becomes a hundred and moves to the left. In the product, the 3 in the tens place of 30 is shifted one place to the left to represent 3 hundreds. In 300 divided by 10 the 3 is shifted one place to the right in the quotient to represent 3 tens.

4.NBT.2 Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using $>$, =, and $<$ symbols to record the results of comparisons.

4.NF.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.[2]

The structure of the base-ten system is uniform



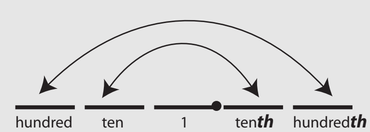
4.NF.4b Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.

b Understand a multiple of a/b as a multiple of $1/b$, and use this understanding to multiply a fraction by a whole number.

4.NF.6 Use decimal notation for fractions with denominators 10 or 100.

4.NF.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, =, or $<$, and justify the conclusions, e.g., by using a visual model.

Symmetry with respect to the ones place



However, because one is the basic unit from which the other base-ten units are derived, the symmetry occurs instead with respect to the ones place.

Ways of reading decimals aloud vary. Mathematicians and scientists often read 0.15 aloud as "zero point one five" or "point one five." (Decimals smaller than one may be written with or without a zero before the decimal point.) Decimals with many non-zero digits are more easily read aloud in this manner. (For example, the number π, which has infinitely many non-zero digits, begins $3.1415\ldots$.)

Other ways to read 0.15 aloud are "1 tenth and 5 hundredths" and "15 hundredths," just as 1,500 is sometimes read "15 hundred" or "1 thousand, 5 hundred." Similarly, 150 is read "one hundred and fifty" or "a hundred fifty" and understood as 15 tens, as 10 tens and 5 tens, and as $100 + 50$.

Just as 15 is understood as 15 ones and as 1 ten and 5 ones in computations with whole numbers, 0.15 is viewed as 15 hundredths and as 1 tenth and 5 hundredths in computations with decimals.

It takes time to develop understanding and fluency with the different forms. Layered cards for decimals can help students become fluent with decimal equivalencies such as three tenths is thirty hundredths.

Use place value understanding and properties of operations to perform multi-digit arithmetic At Grade 4, students become fluent with the standard addition and subtraction algorithms.[4.NBT.4] As discussed at the beginning of this progression, these algorithms rely on adding or subtracting like base-ten units (ones with ones, tens with tens, hundreds with hundreds, and so on) and composing or decomposing base-ten units as needed (such as composing 10 ones to make 1 ten or decomposing 1 hundred to make 10 tens). In mathematics, an algorithm is defined by its steps and not by the way those steps are recorded in writing. With this in mind, minor variations in methods of recording standard algorithms are acceptable.

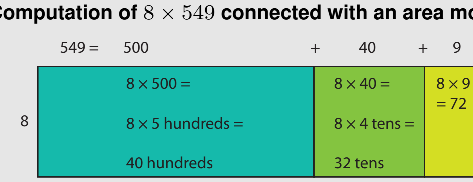
[4.NBT.4] Fluently add and subtract multi-digit whole numbers using the standard algorithm.

In fourth grade, students compute products of one-digit numbers and multi-digit numbers (up to four digits) and products of two two-digit numbers.[4.NBT.5] They divide multi-digit numbers (up to four digits) by one-digit numbers. As with addition and subtraction, students should use methods they understand and can explain. Visual representations such as area and array diagrams that students draw and connect to equations and other written numerical work are useful for this purpose. By reasoning repeatedly about the connection between math drawings and written numerical work, students can come to see multiplication and division algorithms as abbreviations or summaries of their reasoning about quantities.

Students can invent and use fast special strategies while also working towards understanding general methods and the standard algorithm.

[4.NBT.5] Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Computation of 8×549 connected with an area model



Each part of the region above corresponds to one of the terms in the computation below.

$$\begin{aligned} 8 \times 549 &= 8 \times (500 + 40 + 9) \\ &= 8 \times 500 + 8 \times 40 + 8 \times 9. \end{aligned}$$

This can also viewed as finding how many objects are in 8 groups of 549 objects, by finding the cardinalities of 8 groups of 500, 8 groups of 40, and 8 groups of 9, then adding them.

One component of understanding general methods for multiplication is understanding how to compute products of one-digit numbers and multiples of 10, 100, and 1000. This extends work in Grade 3 on products of one-digit numbers and multiples of 10. We can calculate 6×700 by calculating 6×7 and then shifting the result to the left two places (by placing two zeros at the end to show that these are hundreds) because 6 groups of 7 hundred is 6×7 hundreds, which is 42 hundreds, or 4,200. Students can use this place value reasoning, which can also be supported with diagrams of arrays or areas, as they develop and practice using the patterns in relationships among products such as 6×7, 6×70, 6×700, and 6×7000. Products of 5 and even numbers, such as 5×4, 5×40, 5×400, 5×4000 and 4×5, 4×50, 4×500, 4×5000 might be discussed and practiced separately afterwards because they may seem at first to violate the patterns by having an "extra" 0 that comes from the one-digit product.

Another part of understanding general base-ten methods for multi-digit multiplication is understanding the role played by the distributive property. This allows numbers to be decomposed into base-ten units, products of the units to be computed, then combined. By decomposing the factors into like base-ten units and applying the distributive property, multiplication computations are reduced to single-digit multiplications and products of numbers with multiples of 10, of 100, and of 1000. Students can connect diagrams of areas or arrays to numerical work to develop understanding of general base-ten multiplication methods.

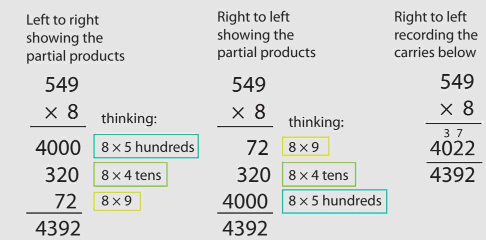
Computing products of two two-digit numbers requires using the distributive property several times when the factors are decomposed into base-ten units. For example,

$$\begin{aligned} 36 \times 94 &= (30+6) \times (90+4) \\ &= (30+6) \times 90 + (30+6) \times 4 \\ &= 30 \times 90 + 6 \times 90 + 30 \times 4 + 6 \times 4. \end{aligned}$$

General methods for computing quotients of multi-digit numbers and one-digit numbers rely on the same understandings as for multiplication, but cast in terms of division.[4.NBT.6] One component is quotients of multiples of 10, 100, or 1000 and one-digit numbers. For example, $42 \div 6$ is related to $420 \div 6$ and $4200 \div 6$. Students can draw on their work with multiplication and they can also reason that $4200 \div 6$ means partitioning 42 hundreds into 6 equal groups, so there are 7 hundreds in each group.

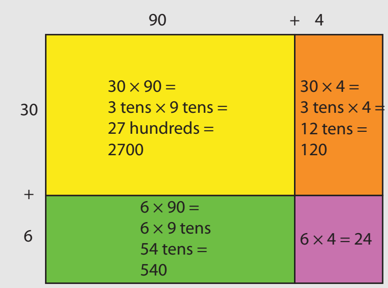
Another component of understanding general methods for multi-digit division computation is the idea of decomposing the dividend into like base-ten units and finding the quotient unit by unit, starting with the largest unit and continuing on to smaller units. As with multiplication, this relies on the distributive property. This can be viewed as finding the side length of a rectangle (the divisor is the length of the other side) or as allocating objects (the divisor is the number of groups). See the figures on the next page for examples.

Computation of 8×549: Ways to record general methods



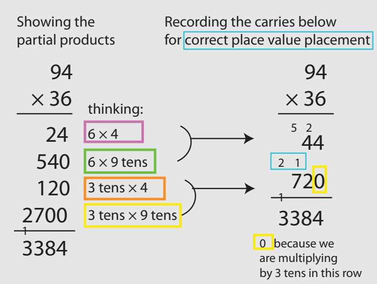
The first method proceeds from left to right, and the others from right to left. In the third method, the digits representing new units are written below the line rather than above 549, thus keeping the digits of the products close to each other, e.g., the 7 from $8 \times 9 = 72$ is written diagonally to the left of the 2 rather than above the 4 in 549.

Computation of 36×94 connected with an area model



The products of like base-ten units are shown as parts of a rectangular region.

Computation of 36×94: Ways to record general methods



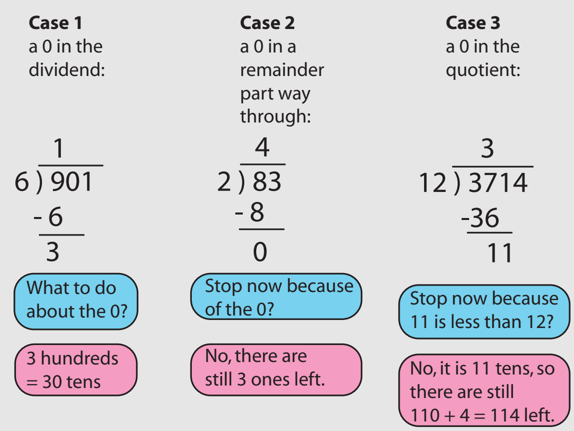
These proceed from right to left, but could go left to right. On the right, digits that represent newly composed tens and hundreds are written below the line instead of above 94. The digits 2 and 1 are surrounded by a blue box. The 1 from $30 \times 4 = 120$ is placed correctly in the hundreds place and the digit 2 from $30 \times 90 = 2700$ is placed correctly in the thousands place. If these digits had been placed above 94, they would be in incorrect places. Note that the 0 (surrounded by a yellow box) in the ones place of the second line of the method on the right is there because the whole line of digits is produced by multiplying by 30 (not 3).

4.NBT.6 Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

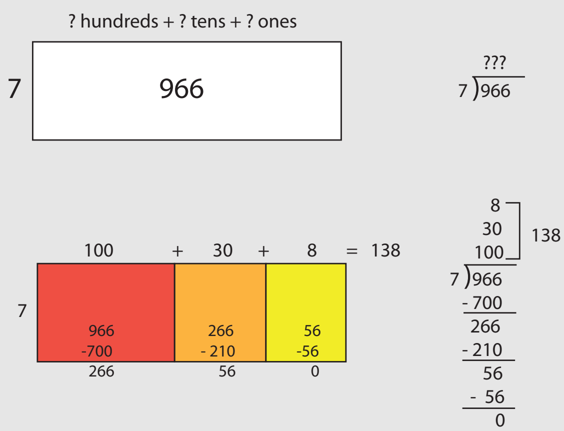
Multi-digit division requires working with remainders. In preparation for working with remainders, students can compute sums of a product and a number, such as $4 \times 8 + 3$. In multi-digit division, students will need to find the greatest multiple less than a given number. For example, when dividing by 6, the greatest multiple of 6 less than 50 is $6 \times 8 = 48$. Students can think of these "greatest multiples" in terms of putting objects into groups. For example, when 50 objects are shared among 6 groups, the largest whole number of objects that can be put in each group is 8, and 2 objects are left over. (Or when 50 objects are allocated into groups of 6, the largest whole number of groups that can be made is 8, and 2 objects are left over.) The equation $6 \times 8 + 2 = 50$ (or $8 \times 6 + 2 = 50$) corresponds with this situation.

Cases involving 0 in division may require special attention.

Cases involving 0 in division

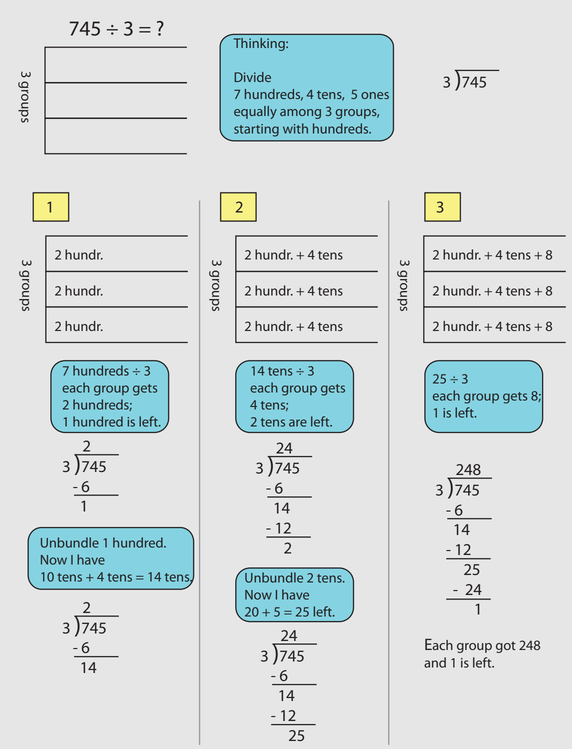


Division as finding side length



$966 \div 7$ is viewed as finding the unknown side length of a rectangular region with area 966 square units and a side of length 7 units. The amount of hundreds is found, then tens, then ones. This yields a decomposition into three regions of dimensions 7 by 100, 7 by 30, and 7 by 8. It can be connected with the decomposition of 966 as $7 \times 100 + 7 \times 30 + 7 \times 8$. By the distributive property, this is $7 \times (100 + 30 + 8)$, so the unknown side length is 138. In the recording on the right, amounts of hundreds, tens, and ones are represented by numbers rather than by digits, e.g., 700 instead of 7.

Division as finding group size



$745 \div 3$ can be viewed as allocating 745 objects bundled in 7 hundreds, 4 tens, and 3 ones equally among 3 groups. In Step 1, the 2 indicates that each group got 2 hundreds, the 6 is the number of hundreds allocated, and the 1 is the number of hundreds not allocated. After Step 1, the remaining hundred is decomposed as 10 tens and combined with the 4 tens (in 745) to make 14 tens.

Grade 5

In Grade 5, students extend their understanding of the base-ten system to decimals to the thousandths place, building on their Grade 4 work with tenths and hundredths. They become fluent with the standard multiplication algorithm with multi-digit whole numbers. They reason about dividing whole numbers with two-digit divisors, and reason about adding, subtracting, multiplying, and dividing decimals to hundredths.

Understand the place value system Students extend their understanding of the base-ten system to the relationship between adjacent places, how numbers compare, and how numbers round for decimals to thousandths.

New at Grade 5 is the use of whole number exponents to denote powers of 10.[5.NBT.2] Students understand why multiplying by a power of 10 shifts the digits of a whole number or decimal that many places to the left. For example, multiplying by 10^4 is multiplying by 10 four times. Multiplying by 10 once shifts every digit of the multiplicand one place to the left in the product (the product is ten times as large) because in the base-ten system the value of each place is 10 times the value of the place to its right. So multiplying by 10 four times shifts every digit 4 places to the left. Patterns in the number of 0s in products of a whole numbers and a power of 10 and the location of the decimal point in products of decimals with powers of 10 can be explained in terms of place value. Because students have developed their understandings of and computations with decimals in terms of multiples (consistent with 4.OA.4) rather than powers, connecting the terminology of multiples with that of powers affords connections between understanding of multiplication and exponentiation.

5.NBT.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.

Perform operations with multi-digit whole numbers and with decimals to hundredths At Grade 5, students fluently compute products of whole numbers using the standard algorithm.[5.NBT.5] Underlying this algorithm are the properties of operations and the base-ten system (see the Grade 4 section).

5.NBT.5 Fluently multiply multi-digit whole numbers using the standard algorithm.

Division strategies in Grade 5 involve breaking the dividend apart into like base-ten units and applying the distributive property to find the quotient place by place, starting from the highest place. (Division can also be viewed as finding an unknown factor: the dividend is the product, the divisor is the known factor, and the quotient is the unknown factor.) Students continue their fourth grade work on division, extending it to computation of whole number quotients with dividends of up to four digits and two-digit divisors. Estimation becomes relevant when extending to two-digit divisors. Even if students round appropriately, the resulting estimate may need to be adjusted.

Recording division after an underestimate

$1655 \div 27$	1
	10 } 61
Rounding 27	(30) 50
to 30 produces	27) 1655
the underestimate	-1350
50 at the first step	305
but this method	-270
allows the division	35
process to be	-27
continued	8

Because of the uniformity of the structure of the base-ten system, students use the same place value understanding for adding and subtracting decimals that they used for adding and subtracting whole numbers.5.NBT.7 Like base-ten units must be added and subtracted, so students need to attend to aligning the corresponding places correctly (this also aligns the decimal points). It can help to put 0s in places so that all numbers show the same number of places to the right of the decimal point. Although whole numbers are not usually written with a decimal point, but that a decimal point with 0s on its right can be inserted (e.g., 16 can also be written as 16.0 or 16.00). The process of composing and decomposing a base-ten unit is the same for decimals as for whole numbers and the same methods of recording numerical work can be used with decimals as with whole numbers. For example, students can write digits representing new units below on the addition or subtraction line, and they can decompose units wherever needed before subtracting.

5.NBT.7 Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

General methods used for computing products of whole numbers extend to products of decimals. Because the expectations for decimals are limited to thousandths and expectations for factors are limited to hundredths at this grade level, students will multiply tenths with tenths and tenths with hundredths, but they need not multiply hundredths with hundredths. Before students consider decimal multiplication more generally, they can study the effect of multiplying by 0.1 and by 0.01 to explain why the product is ten or a hundred times as small as the multiplicand (moves one or two places to the right). They can then extend their reasoning to multipliers that are single-digit multiples of 0.1 and 0.01 (e.g., 0.2 and 0.02, etc.).

There are several lines of reasoning that students can use to explain the placement of the decimal point in other products of decimals. Students can think about the product of the smallest base-ten units of each factor. For example, a tenth times a tenth is a hundredth, so 3.2×7.1 will have an entry in the hundredth place. Note, however, that students might place the decimal point incorrectly for 3.2×8.5 unless they take into account the 0 in the ones place of 32×85. (Or they can think of 0.2×0.5 as 10 hundredths.) They can also think of the decimals as fractions or as whole numbers divided by 10 or 100.5.NF.3 When they place the decimal point in the product, they have to divide by a 10 from each factor or 100 from one factor. For example, to see that $0.6 \times 0.8 = 0.48$, students can use fractions: $\frac{6}{10} \times \frac{8}{10} = \frac{48}{100}$.5.NF.4 Students can also reason that when they carry out the multiplication without the decimal point, they have multiplied each decimal factor by 10 or 100, so they will need to divide by those numbers in the end to get the correct answer. Also, students can use reasoning about the sizes of numbers to determine the placement of the decimal point. For example, 3.2×8.5 should be close to 3×9, so 27.2 is a more reasonable product for 3.2×8.5 than 2.72 or 272. This estimation-based method is not reliable in all cases, however, especially in cases students will encounter in later grades. For example, it is not easy to decide where to place

5.NF.3 Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem.

5.NF.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.

the decimal point in 0.023×0.0045 based on estimation. Students can summarize the results of their reasoning such as those above as specific numerical patterns and then as one general overall pattern such as "the number of decimal places in the product is the sum of the number of decimal places in each factor."

General methods used for computing quotients of whole numbers extend to decimals with the additional issue of placing the decimal point in the quotient. As with decimal multiplication, students can first examine the cases of dividing by 0.1 and 0.01 to see that the quotient becomes 10 times or 100 times as large as the dividend (see also the Number and Operations—Fractions Progression). For example, students can view $7 \div 0.1 = \square$ as asking how many tenths are in 7.[5.NF.7b] Because it takes 10 tenths make 1, it takes 7 times as many tenths to make 7, so $7 \div 0.1 = 7 \times 10 = 70$. Or students could note that 7 is 70 tenths, so asking how many tenths are in 7 is the same as asking how many tenths are in 70 tenths, which is 70. In other words, $7 \div 0.1$ is the same as $70 \div 1$. So dividing by 0.1 moves the number 7 one place to the left, the quotient is ten times as big as the dividend. As with decimal multiplication, students can then proceed to more general cases. For example, to calculate $7 \div 0.2$, students can reason that 0.2 is 2 tenths and 7 is 70 tenths, so asking how many 2 tenths are in 7 is the same as asking how many 2 tenths are in 70 tenths. In other words, $7 \div 0.2$ is the same as $70 \div 2$; multiplying both the 7 and the 0.2 by 10 results in the same quotient. Or students could calculate $7 \div 0.2$ by viewing 0.2 as 2×0.1, so they can first divide 7 by 2, which is 3.5, and then divide that result by 0.1, which makes 3.5 ten times as large, namely 35. Dividing by a decimal less than 1 results in a quotient larger than the dividend[5.NF.5] and moves the digits of the dividend one place to the left. Students can summarize the results of their reasoning as specific numerical patterns then as one general overall pattern such as "when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places."

5.NF.7b Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.[3]

b Interpret division of a whole number by a unit fraction, and compute such quotients.

5.NF.5 Interpret multiplication as scaling (resizing), by:

a Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

b Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

Extending beyond Grade 5

At Grade 6, students extend their fluency with the standard algorithms, using these for all four operations with decimals and to compute quotients of multi-digit numbers. At Grade 6 and beyond, students may occasionally compute with numbers larger than those specified in earlier grades as required for solving problems, but the Standards do not specify mastery with such numbers.

In Grade 6, students extend the base-ten system to negative numbers. In Grade 7, they begin to do arithmetic with such numbers.

By reasoning about the standard division algorithm, students learn in Grade 7 that every fraction can be represented with a decimal that either terminates or repeats. In Grade 8, students learn informally that every number has a decimal expansion, and that those with a terminating or repeating decimal representation are rational numbers (i.e., can be represented as a quotient of integers). There are numbers that are not rational (irrational numbers), such as the square root of 2. (It is not obvious that the square root of 2 is not rational, but this can be proved.) In fact, surprisingly, it turns out that most numbers are not rational. Irrational numbers can always be approximated by rational numbers.

In Grade 8, students build on their work with rounding and exponents when they begin working with scientific notation. This allows them to express approximations of very large and very small numbers compactly by using exponents and generally only approximately by showing only the most significant digits. For example, the Earth's circumference is approximately 40,000,000 m. In scientific notation, this is 4×10^7 m.

The Common Core Standards are designed so that ideas used in base-ten computation, as well as in other domains, can support later learning. For example, use of the distributive property occurs together with the idea of combining like units in the NBT and NF standards. Students use these ideas again when they calculate with polynomials in high school.

The distributive property and like units: Multiplication of whole numbers and polynomials

52×73	$(5x + 2)(7x + 3)$	
$= (5 \times 10 + 2)(7 \times 10 + 3)$	$= (5x + 2)(7x + 3)$	decomposing as like units (powers of 10 or powers of x)
$= 5 \times 10(7 \times 10 + 3) + 2 \times (7 \times 10 + 3)$	$= 5x(7x + 3) + 2(7x + 3)$	using the distributive property
$= 35 \times 10^2 + 15 \times 10 + 14 \times 10 + 2 \times 3$	$= 35x^2 + 15x + 14x + 2 \times 3$	using the distributive property again
$= 35 \times 10^2 + 29 \times 10 + 6$	$= 35x^2 + 29x + 6$	combining like units (powers of 10 or powers of x)